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Effect of Helical Corrugations on the Low Reynolds Number Flow in a Tube

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The low Reynolds number flow in a tube with helically corrugated wavy wall is solved for the first time. The Navier–Stokes equations are perturbed using the small wavy amplitude as parameter. For given pitch λ there exists an optimum number of circumferential waves n where the resistance is minimum. For given n there exists an optimum λ where the induced axial rotation of the flow is maximum. © 2006 American Institute of Chemical Engineers *AIChE J,* 52: 2008–2012, 2006
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Introduction

The well-known Poiseuille resistance formula is adequate for laminar flow in a long, smooth circular tube. However, there are instances where the tube wall may not be smooth but helically corrugated. These helical nonuniformities may be a attributable to manufacturing processes or to promote axial rotation or mixing. There is no previous work on the flow in a helically corrugated tube. Related work includes longitudinally corrugated tubes (corrugations parallel to the tube axis), where the governing simpler Poisson equation can be solved by conformal mapping[1] or small-amplitude perturbation.[2] In the case of transversely corrugated tubes (corrugations perpendicular to the tube axis), the Navier–Stokes equations essentially reduce to a biharmonic equation in terms of a stream function. Methods include approximations by assuming long wavelengths,[3-6] small amplitudes,[7] or numerical integration.[8-14] The present work is the first theoretical attempt to quantify the effect of helical striations on flow in a tube.

We assume small Reynolds numbers and apply perturbation methods by using the small amplitude of the corrugations. Because the flow is three-dimensional, the problem cannot be described solely by the axial velocity, as in the case of longitudinal corrugations, or by the stream function, as in the case of transverse corrugations. The primary variables must be used.

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Formulation

Consider a tube with helical corrugations (Figure 1) described by

$$r' = a + b\sin(n\theta + \lambda z'/a) \tag{1}$$

where (r', θ, z') are orthogonal cylindrical coordinates. The form of Eq. 1 is similar to that of an oblique wave. There are n (integer) circumferential waves and the pitch is $2\pi a/\lambda$. The cross-sectional area is $\pi(a^2 + b^2/2)$, independent of n or λ, both of which are not zero. Let the amplitude be small, such that $\varepsilon = b/a \ll 1$. Let the mean axial pressure gradient be given as G and μ represents the viscosity. We shall use the primary variables in velocities and pressure. The continuity and Navier–Stokes equations become

$$u_r + \frac{u}{r} + \frac{v_\theta}{r} + w_z = 0 \tag{2}$$

$$R\left(uu_r + \frac{vu_\theta}{r} - \frac{v^2}{r} + wu_z\right) = -4p_r + \nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}v_\theta \tag{3}$$

$$R\left(uv_r + \frac{vv_\theta}{r} + \frac{uv}{r} + wv_z\right) = -\frac{4}{r}p_\theta + \nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}u_\theta \tag{4}$$

$$R\left(uw_r + \frac{vw_\theta}{r} + ww_z\right) = -4p_z + \nabla^2 w \tag{5}$$

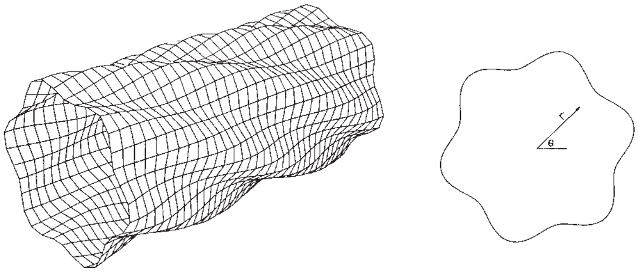

Figure 1. Tube with helical corrugations and a cross section.
$n = 6$, $\lambda = 3$, $\varepsilon = 0.1$.

where (u, v, w) are nondimensional velocity components, that is, velocity components normalized by $Ga^2/(4\mu)$, in the (r, θ, z) directions (lengths are normalized by a), p is the pressure normalized by Ga, and R is the Reynolds number $\rho a^3 G/(4\mu^2)$. The boundary conditions are such that the velocities are finite on the tube axis and zero on the wall at

$$r = 1 + \varepsilon \sin(n\theta + \lambda z) \tag{6}$$

When there are no corrugations ($\varepsilon = 0$) the solution is Poiseuille flow

$$w_0 = 1 - r^2 \qquad u_0 = v_0 = 0 \qquad p_0 = -z \tag{7}$$

We shall perturb from this state:

$$w = 1 - r^2 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots$$

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \qquad v = \varepsilon v_1 + \varepsilon^2 v_2 + \cdots$$

$$p = -z + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots \tag{8}$$

We also assume the Reynolds number is small (small radius or large viscosity) and set $R = \varepsilon k$, where k is of order unity. On the wall, any function f can be expanded about $r = 1$ as follows:

$$f|_{1+\varepsilon S} = f_0|_1 + \varepsilon(f_1|_1 + Sf_{0r}|_1) + \varepsilon^2(f_2|_1 + Sf_{1r}|_1 + S^2 f_{0rr}|_1/2) + \cdots \tag{9}$$

where $S = \sin(n\theta + \lambda z)$. Equations 2–5 are to be solved for successive orders of ε.

The first-order solution

The boundary conditions suggest all variables are functions of r and $\zeta = n\theta + \lambda z$. Thus the first-order terms of Eqs. 2–5 become

$$u_{1r} + \frac{u_1}{r} + \frac{n}{r} v_{1\zeta} + \lambda w_{1\zeta} = 0 \tag{10}$$

$$4p_{1r} = \nabla^2 u_1 - \frac{u_1}{r^2} - \frac{2n}{r^2} v_{1\zeta} \tag{11}$$

$$\frac{4n}{r} p_{1\zeta} = \nabla^2 v_1 - \frac{v_1}{r^2} + \frac{2n}{r^2} u_{1\zeta} \tag{12}$$

$$4\lambda p_{1\zeta} = \nabla^2 w_1 \tag{13}$$

where, under the transform, the 3-D Laplace operator is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \left(\frac{n^2}{r^2} + \lambda^2\right)\frac{\partial^2}{\partial \zeta^2} \tag{14}$$

By use of Eq. 9, the boundary conditions are

$$w_1|_1 = 2\sin\zeta \qquad u_1|_1 = 0 \qquad v_1|_1 = 0 \tag{15}$$

The boundary conditions suggest

$$w_1 = \sin\zeta W(r) \qquad u_1 = \cos\zeta U(r)$$

$$v_1 = \sin\zeta V(r) \qquad p_1 = \cos\zeta P(r) \tag{16}$$

Then Eqs. 10–13 become the coupled ordinary differential equations:

$$U' + \frac{U}{r} + \frac{n}{r} V + \lambda W = 0 \tag{17}$$

$$4P' = LU - \frac{U}{r^2} - \frac{2n}{r^2} V \tag{18}$$

$$-\frac{4n}{r} P = LV - \frac{V}{r^2} - \frac{2n}{r^2} U \tag{19}$$

$$-4\lambda P = LW \tag{20}$$

where Eq. 14 becomes

$$L \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{n^2}{r^2} + \lambda^2\right) \tag{21}$$

Eliminate P from Eqs. 18–20 and use Eq. 17 to obtain

$$(\lambda^2 r^2 + n^2)LW + 2\lambda^2 r W' + \lambda r^2 LU' + \lambda r LU + 2\lambda r U'' + (1 - 2n^2)\lambda U/r = 0 \tag{22}$$

$$(LW)' + 2\lambda^2 W/r + \lambda LU + 2\lambda U'/r + \lambda U/r^2 = 0 \tag{23}$$

Again, after some work, eliminate U to obtain

$$L(LW) = 0 \tag{24}$$

The solution (bounded on the axis) is

$$W = C_1 I_n(\lambda r) + C_2 r I_{n+1}(\lambda r) \tag{25}$$

where C values are constants and I values are modified Bessel functions.[15] Substitution into Eq. 23 gives

$$U'' + \frac{3}{r}U' - \left(\frac{n^2-1}{r^2} + \lambda^2\right)U = -2C_1\lambda I_n/r - 2C_2(2\lambda I_{n+1} + nI_n/r) \tag{26}$$

The solution is

$$U = -C_1 I_{n-1}(\lambda r) - C_2[rI_n(\lambda r) - (n+2)I_{n-1}(\lambda r)/\lambda] + C_3 I_n(\lambda r)/r \tag{27}$$

Equations 17, 25, and 27 yield

$$V = -C_1 I_{n-1}(\lambda r) + C_2(n+2)I_{n-1}(\lambda r)/\lambda + C_3[\lambda I_{n-1}(\lambda r)/n - I_n(\lambda r)/r] \tag{28}$$

The boundary conditions from Eq. 15 are

$$W(1) = 2 \qquad U(1) = 0 \qquad V(1) = 0 \tag{29}$$

from which the coefficients are solved

$$C_1 = 2\{\lambda(n+2)I_{n-1}^2(\lambda) - [\lambda^2 + 2n(n+2)]I_{n-1}(\lambda)I_n(\lambda) + \lambda n I_n^2(\lambda)\}/D \tag{30}$$

$$C_2 = 2\lambda I_{n-1}(\lambda)[\lambda I_{n-1}(\lambda) - 2nI_n(\lambda)]/D \tag{31}$$

$$C_3 = -2\lambda n I_{n-1}(\lambda)I_n(\lambda)/D \tag{32}$$

where

$$D = \lambda I_{n-1}^2(\lambda)[\lambda I_{n-1}(\lambda) + (2-3n)I_n(\lambda)] - I_n^2(\lambda)\{[\lambda^2 - 2n(n-2)]I_{n-1}(\lambda) - \lambda n I_n(\lambda)\} \tag{33}$$

From Eq. 20, the pressure is

$$P = -C_2 I_n(\lambda r)/2 \tag{34}$$

Second-order solution

The first-order solution is periodic and does not contribute directly to the mean flow rate. For that we need the second-order solution, but only the nonperiodic parts. The second-order terms of Eqs. 2–5 are

$$u_{2r} + \frac{u_2}{r} + \frac{n}{r}v_{2\zeta} + \lambda w_{2\zeta} = 0 \tag{35}$$

$$k\lambda w_0 u_{1\zeta} + 4p_{2r} = \nabla^2 u_2 - \frac{u_2}{r^2} - \frac{2n}{r^2}v_{2\zeta} \tag{36}$$

$$k\lambda w_0 v_{1\zeta} + \frac{4n}{r}p_{2\zeta} = \nabla^2 v_2 - \frac{v_2}{r^2} + \frac{2n}{r^2}u_{2\zeta} \tag{37}$$

$$k(u_1 w_{0r} + \lambda w_0 w_{1\zeta}) + 4\lambda p_{2\zeta} = \nabla^2 w_2 \tag{38}$$

Given that $n \neq 0$, let an overbar denote the θ-averaged mean

$$\bar{Q} = \frac{1}{2\pi}\int_0^{2\pi} Q\,d\theta = \frac{1}{2\pi n}\int_{\lambda z}^{2n\pi+\lambda z} Q\,d\zeta \tag{39}$$

Equations 35–38 become

$$\bar{u}_{2r} + \frac{\bar{u}_2}{r} = 0 \tag{40}$$

$$4\bar{p}_{2r} = \bar{u}_2'' + \bar{u}_2'/r - \bar{u}_2/r^2 \tag{41}$$

$$0 = \bar{v}_2'' + \bar{v}_2'/r - \bar{v}_2/r^2 \tag{42}$$

$$0 = \bar{w}_2'' + \bar{w}_2'/r \tag{43}$$

The boundary conditions are

$$\overline{u_2|_1} = -\overline{\sin\zeta u_1|_1} = 0 \tag{44}$$

$$\overline{v_2|_1} = -\overline{\sin\zeta v_1|_1} = -V'(1)/2 \tag{45}$$

$$\overline{w_2|_1} = -\overline{\sin\zeta w_{1r}|_1} - \overline{\sin^2\zeta w_{0rr}|_1} = [1 - W'(1)]/2 \tag{46}$$

The solutions are

$$\bar{u}_2 = 0 \qquad \bar{p}_2 = \text{const.} \tag{47}$$

$$\bar{v}_2 = -V'(1)r/2 \tag{48}$$

$$\bar{w}_2 = [1 - W'(1)]/2 \tag{49}$$

Flow rate and rotation

Let

$$H(x) = \int_0^x wrdr \tag{50}$$

Then from Eq. 9

$$H(1 + \varepsilon\sin\zeta) = \int_0^1 w_0 rdr + \varepsilon\left(\int_0^1 w_1 rdr + \sin\zeta w_0|_1\right) + \varepsilon^2\left[\int_0^1 w_2 rdr + \sin\zeta w_1|_1 + \sin^2\zeta(w_0 r)_r|_1/2\right] + \cdots \tag{51}$$

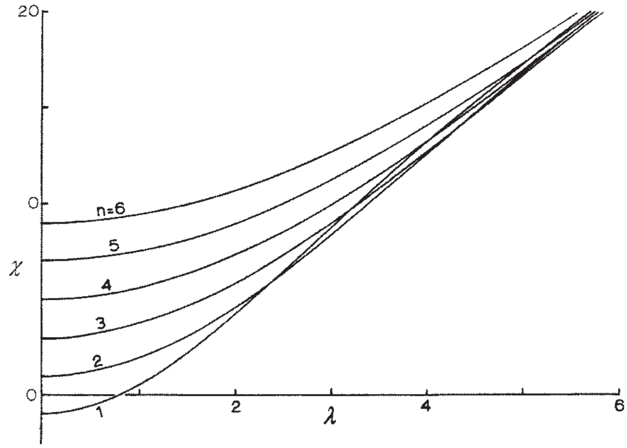

Figure 2. Function χ representing flow rate changes.

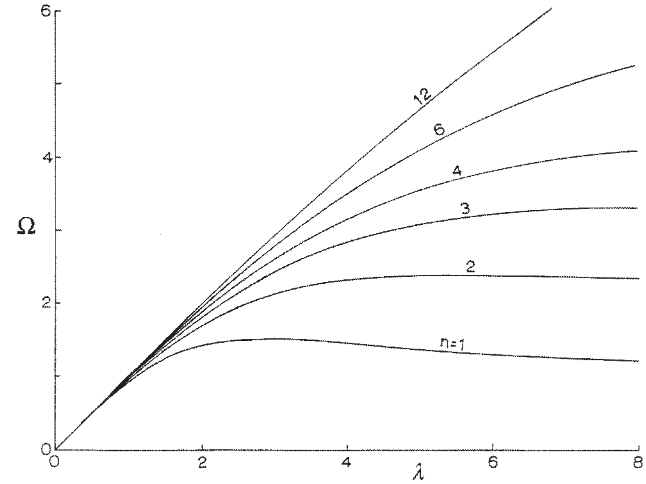

Figure 3. Function Ω representing induced interior rotation.

The flow rate is

$$F = \int_0^{2\pi}\left(\int_0^{1+\varepsilon \sin\zeta} wrdr\right)d\theta = 2\pi\overline{H(1+\varepsilon\sin\zeta)}$$

$$= 2\pi\left\{\int_0^1 w_0rdr + \varepsilon^2\left[\int_0^1 \overline{w_2}rdr + W(1)/2 - 1/2\right] + \cdots\right\}$$

$$= \frac{\pi}{2}[1 - \varepsilon^2\chi + O(\varepsilon^4)] \qquad (52)$$

where

$$\chi = W'(1) - 3 \qquad (53)$$

Figure 2 shows the value of χ, where $\varepsilon^2\chi$ is the fractional decrease in flow. Because the curves intertwine somewhat, the numerical values are given in Table 1.

It is seen that for a given value of n the value of χ increases as λ is increased, whereas for a given value of λ there exists an optimum n where χ is minimum (indicated by the italicized values in Table 1) where flow becomes maximum. Also note χ is negative for low $n = 1$ and low λ, the reason for which is as follows. The normalized cross-sectional area is $2\pi(1 + \varepsilon^2/2)$, larger than the area of a circle of radius 1. For these cases, the increased flow arising from a larger area has not been compensated by the decrease in flow arising from the resistance of the corrugations. Asymptotic expansions of the Bessel functions show $\chi \cong 2n - 3$ as $\lambda \to 0$ and $\chi \cong 4\lambda - 3$ for large λ.

Equation 48 shows there is a mean (rigid) rotation of the interior resulting from the helical corrugations. The normalized angular velocity is $\varepsilon^2\Omega$, where

$$\Omega = -V'(1)/2 \qquad (54)$$

Note that the angle of the flow is order ε^2, much smaller than the pitch of the corrugations. Values of Ω are given in Figure 3 and Table 2.

For a given value of λ the rotation increases with the number of corrugations n. We find $\Omega \cong \lambda$ for small λ and $\Omega \cong n[1 + 3/(2\lambda)]$ for large λ. For a given value of n there exists an optimum pitch λ such that the rotation is maximum. Because induced rotation is of primary interest, the optimum pitch is given in Table 3.

Table 1. Values of χ as a Function of Wavenumber n and Inverse Pitch λ

	n							
λ	1	2	3	4	5	6	12	24
0.1	*−0.985*	1.010	3.008	5.006	7.005	9.004	21.00	45.00
0.5	*−0.627*	1.249	3.187	5.150	7.125	9.170	21.06	45.03
1	*0.468*	1.979	3.739	5.593	7.496	9.426	21.23	45.12
2	*4.262*	4.663	5.824	7.298	8.936	10.67	21.92	45.48
3	8.754	*8.396*	8.913	9.910	11.19	12.65	23.04	46.07
4	13.06	*12.52*	12.60	13.17	14.08	15.23	24.57	46.90
5	17.16	16.70	*16.56*	16.82	17.42	18.28	24.48	47.95
6	21.19	20.83	*20.63*	20.70	21.05	21.67	28.13	49.23
8	29.17	28.97	28.78	*28.69*	28.78	29.06	34.08	52.39
12	45.12	45.04	44.93	44.84	*44.79*	44.81	47.14	60.93
24	93.06	93.04	93.02	92.98	92.95	*92.92*	93.11	98.06

Discussion

Because of the three-dimensional nature of the flow, the primary variables are used. The small-amplitude perturbation enables analytic successive solutions in terms of modified Bessel functions. The low Reynolds numbers affect only the second-order velocities. Higher-order corrections can be obtained, but not pursued.

We find for a given pitch of the corrugations, there exists an optimum wavenumber such that the flow rate is maximized (tube resistance is minimized). For a given wavenumber there exists an optimum pitch such that the bulk axial rotation of the flow is maximized. These optimum geometries do not coincide. The error of both flow rate and rotation rate are of order ε^4.

Our analysis does not include the singular case of $n = 0$ (transverse corrugations) because averaging with respect to θ requires $n \neq 0$. For example, the integration of Eq. 1 would not yield a constant cross-sectional area if n were zero. Our analysis does include the case $\lambda = 0$. The limiting value of $\chi \cong 2n - 3$ as $\lambda \to 0$ agrees with that of Phan-Thien,[2] who studied longitudinal corrugations.

The low Reynolds number flow in a helically corrugated tube is now solved. The results can be applied to very viscous fluids, such as syrup, or very small tubes. Using the velocity field, the effects of heat and mass transfer can also be investigated, although the analyses are not included in this article.

Table 2. Values of Ω as a Function of Wavenumber *n* and Inverse Pitch λ

	n							
λ	1	2	3	4	5	6	12	24
0.1	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
0.5	0.490	0.495	0.497	0.498	0.499	0.499	0.500	0.500
1	0.918	0.959	0.976	0.984	0.988	0.991	0.997	0.999
2	1.422	1.703	1.816	1.875	1.909	1.931	1.978	1.994
3	1.509	2.140	2.442	2.609	2.712	2.779	2.928	2.979
4	1.443	2.332	2.849	3.165	3.370	3.509	3.834	3.952
5	1.363	2.389	3.087	3.557	3.882	4.112	4.685	4.906
6	1.299	2.387	3.213	3.820	4.265	4.595	5.474	5.840
8	1.217	2.336	3.300	4.097	4.743	5.262	6.853	7.631
12	1.138	2.242	3.283	4.240	5.105	5.874	8.850	10.85
24	1.066	2.125	3.171	4.198	5.200	6.172	11.23	17.34

Table 3. Optimum Pitch for Maximum Rotation

	n							
	1	2	3	4	5	6	12	24
Ω_{max}	1.511	2.392	3.306	4.247	5.206	6.176	12.09	24.05
λ_{opt}	2.83	5.40	9.12	13.85	19.96	27.46	99.52	388.0

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